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Explicit solutions for a class of nonlinear BSDEs and their nodal sets[#]

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Dedicated to Professor Alain Bensoussan on the occasion of his 80th birthday.

Abstract In this paper, we investigate a class of nonlinear backward stochastic differential equations (BSDEs) arising from financial economics, and give the sign of corresponding solution Z. Furthermore, we are able to obtain explicit solutions to an interesting class of nonlinear BSDEs, including the k-ignorance BSDE arising from the modeling of ambiguity of asset pricing. Moreover, we show its applications in PDEs and contingent pricing in an incomplete market.

Keywords Explicit solution, Feynman–Kac formula, Girsanov's formula, Nodal set, Nonlinear BSDE, Parabolic equation, Tanaka's formula

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1. Introduction

It is well known that in the seminal paper [17], Pardoux and Peng studied nonlinear backward stochastic differential equations (BSDEs)

$$\mathrm{d}Y_t = -g(t, Y_t, Z_t)\mathrm{d}t + Z_t \mathrm{d}B_t, \quad Y_T = \xi,$$

where B is a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $T > 0$ and ξ is integrable. They proved, under some assumptions on the nonlinear driver g and the terminal value ξ, that BSDE (1.1) has a unique solution pair of adapted processes Y and Z satisfying stochastic integral equation:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \quad 0 \le t \le T. \tag{1.1}$$

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Since the publication of [17], many researchers have worked on the theory of BSDEs and have obtained many results about the solution pair (Y_t, Z_t). The theory of BSDEs has been applied to mathematical finance, stochastic control, partial differential equations, stochastic game, and so on, see, for example, [5, 8, 10, 13, 14, 19] and the literature therein.

In mathematical finance and economics, the solution (Z_t) usually represents the amount of risk assets (see [10] for details). On the other hand, (Z_t) is related to the optimal control in stochastic control, see [22] for example. So it is important to have an explicit expression for the solution of BSDE (1.1). Moreover, the sign of solution (Z_t) is also important for models in mathematical finance, which allows us to identify the monotone ranges of active hedging. Furthermore, it can help obtain the explicit solution of particular nonlinear BSDEs when $g(z) = k|z|$. Therefore, researchers are interested in determining the zeros of (Z_t) for such BSDE models, i.e., the nodal set of the process (Z_t).

In the present paper, we study the Markovian case of BSDE (1.1). That is, the driver in formulating the BSDE is a deterministic real function $g(t, y, z)$ for $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, the terminal value is $\xi = \varphi(B_T)$. We show the sign of (Z_t) for these kinds of Markovian BSDEs. Furthermore, we give the explicit solutions of the following nonlinear Markovian BSDE when $g(t, y, z) = k|z|$, which is the k-ignorance model arising from modeling the ambiguity of asset pricing (see Chen and Epstein [5] for details):

$$Y_t = \varphi(B_T) + \int_t^T k|Z_s|\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \tag{1.2}$$

where $k > 0$ is a constant. Note that BSDE (1.2) is perhaps the simplest nonlinear BSDE, but its explicit solution is important in many fields. For example, the explicit solution of BSDE (1.2) can give the explicit representation of the min-max price of the contingent claim ξ. For this aspect, the reader can also refer to Chen and Epstein [5] and the literature therein. Additionally, BSDE (1.2) is related to the study of the nonlinear central limit theorem. A recent work of Chen and Epstein [6] obtains a new nonlinear central limit theorem. They prove that the limit of a sequence of random variables (whose joint distribution is described by a set of measures) is defined by the nonlinear BSDE (1.2). Thus, it is important to get the explicit solution of BSDE (1.2) so that the central limit theorem proposed by Chen and Epstein [6] has an explicit limit.

Though explicit solutions to BSDE (1.1) are important in many fields, they are known only in few cases, mainly in the case where $g(t, y, z)$ is linear in y and z. For a nonlinear driver $g(t, y, z)$, little is known about (Y_t, Z_t) due to the lack of an explicit formula. Chen et al. [4, 7] show that if $\xi = \varphi(B_T)$ and φ is monotonic, then the solution pair (Y, Z) of BSDE (1.2) can be computed explicitly. In their paper, they observe that BSDE (1.2) can be reduced to an equivalent linear BSDE, so that an explicit formula is obtained according to Girsanov's formula. If φ is not monotonic, it remains open to find the explicit solution to BSDE (1.2).

The main difficulty in this paper is that the driver $g(z) = k|z|$ in BSDE (1.2) is nonlinear in z and is only Lipschitz continuous but not differentiable. We cannot use the Feynman-Kac's formula in [18] directly to analyze this nonlinear BSDE, since in their paper the solution of the corresponding partial differential equation (PDE) in this case is the viscosity solution. Moreover, since few works can be referred to when φ is not monotonic, we don't know whether it can be reduced to a linear BSDE. To overcome this difficulty, we use the property of weak solution to quasi-linear parabolic equations and establish the Feynmen-Kac formula for the nonlinear BSDE (1.2). First, by using Feynman-Kac's formula [18] and analyzing the corresponding PDE, we obtain the sign of the solution (Z_t) of BSDE (1.1) in the Markovian case when $\xi = \varphi(B_T)$ and

φ is a symmetric function satisfying some regular conditions. Second, using Feynman-Kac's formula established for BSDE (1.2), we explore the sign of (Z_t) when $g(z)=k|z|$ through analyzing the corresponding PDE to BSDE (1.2). Thus, using Tanaka's formula, we finally give an explicit representation of the solution (Y_t, Z_t) for the nonlinear BSDE (1.2) (see Theorem 3.4 in the following).

As applications, we give the closed-form (see (3.26) and (3.27) in Corollary 3.2) of the explicit solutions of BSDE (1.2) when $\varphi(x)=I_{[a,b)}(x)$ with $a,b\in(-\infty,\infty)$. We point out that when $\varphi(x)=I_{[a,b)}(x)$, the explicit solution Y_0 of BSDE (1.2) is the maximal distribution of the semi-martingale $(B_T+\int_0^T \mu_s \mathrm{d}s)$ over the set $\{-k\le\mu_t\le k\}$, i.e.,

$$Y_0=\max_{\{-k\le\mu_s\le k\}} P\left(a\le B_T+\int_0^T \mu_s \mathrm{d}s<b\right),$$

where μ_s is $\mathcal{F}_s$-adapted process. Moreover, we give an example of the explicit solution of BSDE (1.2) when $\varphi(x)=x^2$. In this case, the solution Y_t is related to the bounded-velocity control problem proposed and also solved by Beneš et al. [2].

The paper proceeds as follows. In section 2, we briefly introduce notation and a few basic facts about BSDEs, which will be used throughout the paper. In section 3, we give the main results of the paper. First, in subsection 3.1, we study the sign of (Z_t) for nonlinear BSDEs (1.1) in the Markovian case. Moreover, we establish a Feynman-Kac formula for k-ignorance BSDE (1.2) and show its sign of Z. Second, in subsection 3.2, we give the explicit solution of BSDE (1.2) when φ is a symmetric function, especially for $\varphi(x)=I_{[a,b]}(x)$. In section 4, we first show its applications to BSDEs and PDEs. Then we apply the explicit solutions to show the maximal and minimal contingent price in an incomplete market.

2. Preliminaries

In this section, we briefly recall some basic results on BSDEs and establish notation we will use. Let $(B_t)_{t\ge0}$ be a standard one-dimensional Brownian motion on a probability space $(\Omega,\mathcal{F},P)$, let $(\mathcal{F}_t)$ be the σ-filtration generated by the Brownian motion, that is, $\mathcal{F}_t=\sigma\{B_s;0\le s\le t\}$ for $t\ge0$.

The driver in formulating the BSDE in this paper is a deterministic real function $g(t,y,z)$ for $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}$, which satisfies the following conditions:

(A.1) *Lipschitz condition*. There exists a constant $k\ge0$, such that

$$|g(t,y_1,z_1)-g(t,y_2,z_2)|\le k(|y_1-y_2|+|z_1-z_2|) \tag{2.1}$$

for all $t\ge0$, $y_1,y_2\in\mathbb{R}$, and $z_1,z_2\in\mathbb{R}$.

We use the standard notation that $L^2(\Omega,\mathcal{F}_t,P)$ denotes the space of $\mathcal{F}_t$-measurable and square-integrable random variables on $(\Omega,\mathcal{F},P)$ for each $t\ge0$. Set

$$\mathcal{M}(0,T,\mathbb{R}):=\left\{(v_t)_{t\in[0,T]}:\ \text{real valued } (\mathcal{F}_t)\text{-adapted process with } E\left[\int_0^T |v_t|^2\mathrm{d}t\right]<\infty\right\}.$$

The fundamental result obtained in Pardoux–Peng [17] is the following. If g satisfies (A.1), and $\xi\in L^2(\Omega,\mathcal{F}_T,P)$, there is a unique pair of adapted processes $Y,Z\in\mathcal{M}(0,T,\mathbb{R})$ which solves BSDE (1.1). We are interested in the Markovian case, that is, the terminal value ξ in BSDE (1.1) depends only on B_T, i.e.,

$$Y_t = \varphi(B_T) + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s. \tag{2.2}$$

In the following, I_A represents the indicator functions on event A; $\mathbb{E}_P[\cdot]$ denotes the expectation under probability measure P, $\mathbb{E}_Q[\cdot]$ is the expectation under probability measure Q; the function $\operatorname{sgn}(x)$ we use is defined by

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ -1, & x \le 0. \end{cases}$$

3. Main results

In this section, we establish main results. In the first subsection, we show the sign of (Z_t) for BSDE (2.2) under some regular conditions on φ and g. Through the same method, we obtain the sign of (Z_t) when the driver is $g = k|z|$ by establishing the Feynman–Kac formula for this particular diver. In the second subsection, using the sign of (Z_t) and Tanaka's formula, we give the explicit solutions of BSDE (2.2) when $g(z) = k|z|$. Assume that φ used in our main result satisfies the following condition:

(H.1) There is $c \in \mathbb{R}$, φ is symmetric about c, that is, $\varphi(c - x) = \varphi(c + x)$ for all $x \in \mathbb{R}$.

3.1 The sign of (Z) for nonlinear Markovian BSDEs

Here, we prove the first result of the paper, which regards the sign of (Z_t) for BSDE (2.2). It can also be stated as a "non-vanishing theorem," which is useful in our discussions in the latter part of the paper.

Theorem 3.1 *Let (Y_t, Z_t) be the unique solution pair of BSDE* (2.2). *Assume that*

(i) *$\varphi \in C^3(\mathbb{R})$ satisfies (H.1) for some $c \in \mathbb{R}$, and $\varphi^{(i)}$ (where $i = 0, 1, 2, 3$) have at most polynomial growth;*

(ii) *$g \in C_b^{1,3}(\mathbb{R}_+ \times \mathbb{R}^2)$ satisfies (A.1), and $g(t, y, \cdot)$ is symmetric at 0, that is, $g(t, y, z) = g(t, y, -z)$ for all $t \ge 0$, y, $z \in \mathbb{R}$.*

Then Y_t, Z_t are continuous adapted processes and the following conclusions hold.

(1) *If $\varphi'(x) \ge 0$ and $\varphi'(x) \not\equiv 0$ for all $x > c$, then $\operatorname{sgn}(Z_t) = \operatorname{sgn}(B_t - c)$ for all $t \ge 0$ almost surely;*

(2) *Similarly, if $\varphi'(x) \le 0$ and $\varphi'(x) \not\equiv 0$ for all $x > c$, then $\operatorname{sgn}(-Z_t) = \operatorname{sgn}(B_t - c)$ for all $t \ge 0$ almost surely.*

Remark 3.1 *While conditions imposed on φ and $g(t, y, z)$ in Theorem 3.1 seem restrictive in applications, the approach put forward can also be applied to other situations where the regularity on the driver $g(t, y, z)$ is not available for applying Theorem 3.1 directly. These instances, however, must be treated on a case-by-case basis (see, e.g., Corollary 3.2).*

Remark 3.2 *Theorem 3.1, similarly, can be generalized to the following BSDE in high dimension,*

$$Y_t = \varphi(B_T^1, \cdots, B_T^d) + \int_t^T \sum_{i=1}^d k_i |Z_s^i| \mathrm{d}s - \int_t^T \sum_{i=1}^d Z_s^i \mathrm{d}B_s^i,$$

where $\varphi(x_1, \cdots, x_d)$ *is symmetric with respect to* $x_1 = c_1, \cdots, x_d = c_d$.

Theorem 3.1 can also be stated as the following "non-vanishing theorem," which relates to the nodal sets of (Z_t).

Theorem 3.2 (**"Non-vanishing Theorem"**) *Suppose* (Y_t, Z_t) *is the unique solution pair of BSDE (2.2), where* g *and* φ *are under the same conditions of Theorem 3.1. Then* $Z \neq 0$ *with respect to the product measure* $\mathrm{d}t \otimes \mathrm{d}P$.

Proof Since $\{B \neq c\}$ almost surely with respect to $\mathrm{d}t \otimes \mathrm{d}P$, the conclusion is a direct consequence of Theorem 3.1 as $\{Z \neq 0\} = \{B \neq c\}$ almost surely. □

In order to prove Theorem 3.1, we first show a lemma.

Lemma 3.1 *Let* $g \in C_b^{1,3}(\mathbb{R}_+ \times \mathbb{R}^2)$ *satisfy (A.1). Assume that* $\varphi \in C^3(\mathbb{R})$ *and the derivatives* $\varphi^{(i)}$ *(where* $i = 0, 1, 2, 3$ *) have at most polynomial growth.*

(1) Let $u(t, x)$ be the unique solution of Cauchy's initial problem of the parabolic equation

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_{xx}^2 u + g(t, u, \partial_x u), \quad \text{in } (0, T] \times \mathbb{R}, \\ u(0, x) = \varphi(x). \end{cases} \tag{3.1}$$

Then $Y_t = u(T - t, B_t)$ and $Z_t = \partial_x u(T - t, B_t)$ are unique solutions of BSDE (2.2).

(2) If in addition φ satisfies (H.1), $g(t, y, z) = g(t, y, -z)$ for any $t \in (0, T]$ and $y, z \in \mathbb{R}$, then we have the following conclusions.

(i) $\partial_x u(t, c) = 0$ for every $t \in (0, T]$.

(ii) $w(t, x) := \partial_x u(t, x)$ is the unique solution to the initial value problem of the parabolic equation

$$\begin{cases} \partial_t w = \frac{1}{2}\partial_{xx}^2 w + \partial_z g(t, u, w) \cdot \partial_x w + \partial_y g(t, u, w) \cdot w, \quad \text{in } (0, T] \times \mathbb{R}, \\ w(0, x) = \varphi'(x). \end{cases} \tag{3.2}$$

Moreover, $w(t, c) = 0$ for $t \in (0, T]$.

(iii) For fixed $x \in \mathbb{R}$ and $t \in (0, T]$, let

$$\begin{aligned} a_{s,t} &:= \partial_y g\left(t - s, u(t - s, X_s^x), w(t - s, X_s^x)\right), \\ b_{s,t} &:= \partial_z g\left(t - s, u(t - s, X_s^x), w(t - s, X_s^x)\right) \end{aligned}$$

for $0 \le s \le t$, where $X_s^x = x + B_s$. Define the stochastic exponential martingale

$$N_s = \exp\left\{\int_0^s b_{r,t}\mathrm{d}B_r - \frac{1}{2}\int_0^s b_{r,t}^2 \mathrm{d}r\right\}, \quad 0 \le s \le t. \tag{3.3}$$

Then

$$w(t, x) = \mathbb{E}\left[N_t \varphi'(X_t^x) \cdot e^{\int_0^t a_{s,t}\mathrm{d}s} I_{\{t<\tau\}}\right] \tag{3.4}$$

for every $t > 0$, where $\tau := \inf\{s \ge 0,\ X_s^x = c\}$.

Proof Since φ is a C^3-function with polynomial growth and $g \in C^{1,3}$, so by the theory of parabolic equations of second order, PDE (3.1) possesses a unique solution $u(t, x)$ which belongs to $C^{1,3}((0, T] \times \mathbb{R})$, see, for example, [11].

By Itô's formula, we know that $Y_t = u(T - t, B_t)$, $Z_t = \partial_x u(T - t, B_t)$ solve BSDE (2.2). Then the conclusion of (1) follows directly from the uniqueness of the solution to BSDE (2.2).

Now, we prove (2). Since $g(t, y, \cdot)$ is symmetric about 0, one can verify that $u(t, c - x)$ and

$u(t, x+c)$ are solutions to the parabolic equation

$$\partial_t v = \frac{1}{2}\partial^2_{xx} v + g(t, v, \partial_x v), \quad \text{in } (0, T] \times \mathbb{R}, \tag{3.5}$$

subject to initial value $\varphi(c-x)$ and $\varphi(c+x)$, respectively. Since φ is symmetric about c, that is, $\varphi(c-x) = \varphi(c+x)$, by the uniqueness of the initial value problem for the parabolic equation (3.5) we can conclude that $u(t, c+x) = u(t, c-x)$. This in turn yields that $\partial_x u(t, c+x) = -\partial_x u(t, c-x)$ for $(t, x) \in (0, T] \times \mathbb{R}$. In particular, $\partial_x u(t, c) = 0$ for every $t \in (0, T]$. We thus prove (i).

(ii) follows immediately by differentiating the parabolic equation (3.1) in x.

(iii) Under assumptions on $g(t, y, z)$, $a_{s,t}$ and $b_{s,t}$ are bounded processes for $0 < s \leq t \leq T$. Since $\varphi \in C^3$ and $\varphi^{(i)}$ ($i = 0, 1, 2, 3$) have at most polynomial growth, the unique strong solution $u(t, x)$ to problem (3.1) belongs to $C^{1,3}((0, T] \times \mathbb{R})$. In particular, we have that $w(t, x) \in C^{1,2}((0, T] \times \mathbb{R})$.

Let us first consider the case where $|\varphi^{(i)}| \leq C$ ($i = 0, 1, 2, 3$). We know in this case the second-order derivative of $u(t, x)$ is bounded uniformly, that is, $|\partial_x w(t, x)| \leq C$ for $(0, T] \times \mathbb{R}$.

For any fixed $0 < t \leq T$, define $q(s)$ by solving the ordinary differential equation

$$\mathrm{d}q(s) = a_{s,t} q(s) \mathrm{d}s, \quad q(0) = 1.$$

Then $q(s)$ is a bounded process which has finite variation. Let N_s be the solution to the following SDE:

$$\mathrm{d}N_s = N_s b_{s,t} \mathrm{d}B_s, \quad N_0 = 1.$$

Then N is the stochastic exponential of $\int_0^\cdot b_{r,t} \mathrm{d}B_r$, where $(b_{s,t})_{s \leq t}$ is a bounded process (while its bound may depend on t).

Denote $M_s := q(s) N_s w(t-s, X_s^x)$, where $0 \leq s \leq t$. By Itô's formula we have

$$\begin{aligned}
\mathrm{d}M_s &= q(s) \mathrm{d}\left[N_s w(t-s, X_s^x)\right] + N_s w(t-s, X_s^x) a_{s,t} q(s) \mathrm{d}s \\
&= q(s) N_s b_{s,t} w(t-s, X_s^x) \mathrm{d}B_s + q(s) N_s \cdot \mathrm{d}w(t-s, X_s^x) \\
&\quad + q(s) N_s b_{s,t} \partial_x w(t-s, X_s^x) \mathrm{d}s + q(s) N_s a_{s,t} w(t-s, X_s^x) \mathrm{d}s.
\end{aligned} \tag{3.6}$$

Since $w(t, x)$ solves (3.2),

$$\begin{aligned}
&\mathrm{d}w(t-s, X_s^x) \\
&= \left(-\partial_s w(t-s, X_s^x) + \frac{1}{2}\partial^2_{xx} w(t-s, X_s^x)\right) \mathrm{d}s + \partial_x w(t-s, X_s^x) \mathrm{d}B_s \\
&= \left(-a_{s,t} w(t-s, X_s^x) - b_{s,t} \partial_x w(t-s, X_s^x)\right) \mathrm{d}s + \partial_x w(t-s, X_s^x) \mathrm{d}B_s.
\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain that

$$\mathrm{d}M_s = q(s) N_s \left[\partial_x w(t-s, X_s^x) + b_{s,t} w(t-s, X_s^x)\right] \mathrm{d}B_s.$$

We claim that M is a square-integrable martingale. In fact, we have

$$\left|q(s) N_s \left[\partial_x w(t-s, X_s^x) + b_{s,t} w(t-s, X_s^x)\right]\right| \leq C_1 N_s$$

for some positive constant C_1 depending on t but not on $s \leq t$. Moreover,

$$\begin{aligned}
\mathbb{E}\left[|N_s|^2\right] &= \mathbb{E}\left[\exp\left(2\int_0^s b_{r,t} \mathrm{d}B_r - \int_0^s |b_{r,t}|^2 \mathrm{d}r\right)\right] \\
&\leq C_2 \mathbb{E}\left[\exp\left(2\int_0^s b_{r,t} \mathrm{d}B_r - 2\int_0^t |b_{r,t}|^2 \mathrm{d}r\right)\right] \\
&= C_2 < \infty,
\end{aligned}$$

where C_2 is a positive constant. Therefore,

$$\begin{aligned}\mathbb{E}\left[|M_t|^2\right] &= \mathbb{E}\left(M_0 + \int_0^t \left[q(s)N_s\partial_x w(t-s, X_s^x) + b_{s,t}w(t-s, X_s^x)\right]\mathrm{d}B_s\right)^2 \\ &\le C_0 + 2\mathbb{E}\left(\int_0^t \left[q(s)N_s\partial_x w(t-s, X_s^x) + b_{s,t}w(t-s, X_s^x)\right]\mathrm{d}B_s\right)^2 \\ &\le C_0 + 2C_1^2\mathbb{E}\left(\int_0^t N_s^2\mathrm{d}s\right) = C_0 + 2C_1^2\int_0^t \mathbb{E}(N_s^2)\mathrm{d}s < \infty,\end{aligned}$$

which implies that (M_s) is a square-integrable martingale up to time t.

Setting

$$\tau := \inf\{s \ge 0,\ X_s^x = c\} = \inf\{s \ge 0,\ B_s = c - x\},$$

then τ is a stopping time and finite almost surely by (2.6) in [12]. According to the stopping theorem for martingales, we have $\mathbb{E}(M_0) = \mathbb{E}(M_{t\wedge\tau})$. Then

$$\begin{aligned}w(t,x) &= \mathbb{E}\left(q(t\wedge\tau)N_{t\wedge\tau}w(t-t\wedge\tau, X_{t\wedge\tau}^x)\right) \\ &= \mathbb{E}\left[N_t\varphi'(X_t^x)e^{\int_0^t a_{r,t}\mathrm{d}r}\cdot I_{\{t<\tau\}}\right] + \mathbb{E}\left[q(\tau)N_\tau w(t-\tau, X_\tau^x)\cdot I_{\{\tau\le t\}}\right] \\ &= \mathbb{E}\left[N_t\varphi'(X_t^x)e^{\int_0^t a_{r,t}\mathrm{d}r}\cdot I_{\{t<\tau\}}\right],\end{aligned}$$

where in the last equality we apply $w(s,c) = 0$ for all $s \in (0,T]$.

A simple approximation procedure allows us to validate the representation for the case where $\varphi^{(i)}$ ($i = 0,1,2,3$) possess polynomial growth. The proof is complete. □

Remark 3.3 *The representation (3.4) is basically Feynman-Kac's formula for the stopped Brownian motion (killed at hitting the level c), together with the Cameron–Martin formula, see Pinsky [20] for more information.*

Remark 3.4 *From the proof of Lemma 3.1, we can see that (H.1) is not mandatory and we only need that there is a constant $c \in R$ such that $w(t,c) = 0$ and φ is monotone on $[c,\infty)$. The symmetric function in (H.1) naturally satisfies the condition $w(t,c) = 0$.*

Remark 3.5 *The BSDE (1.2) is associated with the parabolic equation*

$$\partial_t u = \frac{1}{2}\partial_{xx}u + g(t,u,\partial_x u), \quad u(0,x) = \varphi(x). \tag{3.8}$$

The study of the sign of Z_t is actually equivalent to the study the nodal set of $\partial_x u$. It has a connection to the work of Qian and Xu (2018). For more details, see [21].

Now, we begin to prove Theorem 3.1. The proof is in a routine by Lemma 3.1.

Proof of Theorem 3.1 By Lemma 3.1, $Z_t = w(T-t, B_t)$ and

$$w(t,x) = \mathbb{E}\left[N_t\varphi'(X_t^x)e^{\int_0^t a_{s,t}\mathrm{d}s}\cdot I_{\{t<\tau\}}\right], \tag{3.9}$$

which allows us to determine the sign of $w(t,x)$ accordingly.

Note that when $x > c$, it holds that $X_t^x > c$ on $\{t < \tau\}$. It is obvious that $N_t > 0$ for $t \in (0,T]$. Thus, if $\varphi'(x) \ge 0$ and $\varphi'(x) \not\equiv 0$ for all $x > c$, we have

$$P\left(N_t\varphi'(X_t^x)I_{\{t<\tau\}} > 0\right) > 0,$$

which, combined with (3.9), indicates that $w(t,x) > 0$ for $x > c$. Since $\varphi(x)$ is symmetric about $x = c$, we can obtain in the same way that $w(t,x) < 0$ for $x < c$. Thus, due to that

$Z_t = w(T-t, B_t)$, we have

$$\operatorname{sgn}(Z_t) = \operatorname{sgn}(B_t - c).$$

Similarly, if $\varphi'(x) \le 0$ and $\varphi'(x) \not\equiv 0$ for all $x > c$, we have $\operatorname{sgn}(Z_t) = -\operatorname{sgn}(B_t - c)$.

The proof of Theorem 3.1 is completed. □

In the following, we analyze the sign of (Z_t) for the special and important nonlinear BSDE (1.2). Recall that the driver in BSDE (1.2) is $g(z) = k|z|$, which is nonlinear and Lipschitz continuous in z. Thus, it has a unique solution pair (Y, Z) according to Pardoux–Peng [17]. However, we cannot apply the nonlinear Feynman-Kac formula directly because $g(z) = k|z|$ does not satisfy the derivative conditions in Peng [18]. Thus, the method in Theorem 3.1 cannot be applied to BSDE (1.2) directly. Moreover, the solution $u(t,x)$ to the corresponding parabolic equation is only C^{1+}, but not C^2 in the variable x. Therefore, the first step is to derive a nonlinear Feynman-Kac-type formula for this case, and generalize the results in Theorem 3.1 and Lemma 3.1 to BSDE (1.2).

Theorem 3.3 *Let $\varphi \in C^3(\mathbb{R})$ satisfy (H.1) with some constant c, such that φ and φ' have at most polynomial growth, and let u be the unique weak solution to the nonlinear parabolic equation*

$$\begin{cases} \partial_t u = \dfrac{1}{2}\partial_{xx}^2 u + k|\partial_x u|, \\ u(0,x) = \varphi(x). \end{cases} \tag{3.10}$$

Then $\partial_x u(t,x)$ is Hölder continuous in any compact subset of $(0,\infty)\times\mathbb{R}$; and for every $(t,x) \in (0,\infty)\times\mathbb{R}$, it holds that

$$\partial_x u(t,x) = \mathbb{E}\left[N_t \varphi'(B_t + x)\cdot I_{\{t<\tau\}}\right], \tag{3.11}$$

where

$$N_s = \exp\left[k\int_0^s \operatorname{sgn}(w(t-r, B_r + x))\mathrm{d}B_r - \frac{k^2}{2}s\right]$$

is a martingale for $0 \le s \le t$, $\tau = \inf\{s \ge 0 : B_s + x = c\}$, and $w(t,x) = \partial_x u(t,x)$ is the unique weak solution to the following parabolic equation

$$\begin{cases} \partial_t w = \dfrac{1}{2}\partial_{xx}^2 w + k\cdot\operatorname{sgn}(w(t,x))\cdot\partial_x w, \\ w(0,x) = \varphi'(x). \end{cases} \tag{3.12}$$

Moreover, $Y_t = u(T-t, B_t)$ and $Z_t = w(T-t, B_t)$ is the unique solution pair to BSDE (1.2).

Proof According to the theory of parabolic equations (see section 1, Chapter V in[15], or Theorem 10 on page 72 in [11]), there is a unique weak solution $u(t,x)$ to the problem (3.10) and $\partial_x u(t,x)$ is Hölder continuous on any compact subset of $(0,T)\times\mathbb{R}$. By the following Lemma 3.2, the linear parabolic equation (3.12) has a unique weak solution which is Hölder continuous in any compact set of $(0,T)\times\mathbb{R}$.

Next, we prove that $Y_t = u(T-t, B_t)$ and $Z_t = w(T-t, B_t)$ are unique solutions of BSDE (1.2). To this end, let $g_\varepsilon(z) = k\sqrt{z^2+\varepsilon}$ for $\varepsilon \ge 0$. Then, g_ε is smooth and $|g_\varepsilon(z) - g_0(z)| \to 0$ as $\varepsilon \to 0$ for every $z \in \mathbb{R}$. Moreover, $g'_\varepsilon(z) = k\dfrac{z}{\sqrt{z^2+\varepsilon}}$ so that $|g'_\varepsilon(z)| \le k$.

Since φ has at most polynomial growth, there exists a unique strong solution $u^\varepsilon(t,x)$ to the problem

$$\begin{cases} \partial_t u^\varepsilon(x,t) = \dfrac{1}{2}\partial^2_{xx} u^\varepsilon(t,x) + g_\varepsilon(\partial_x u^\varepsilon(t,x)), \\ u^\varepsilon(x,0) = \varphi(x), \end{cases} \tag{3.13}$$

for every $\varepsilon > 0$. Due to the regularity theory of quasi-linear parabolic equations (see [15]), $u^\varepsilon(t,x) \in C^{1,\infty}((0,\infty)\times\mathbb{R})$, and its space derivative $w^\varepsilon(t,x) := \partial_x u^\varepsilon(t,x)$ is the unique weak solution to the (linear) parabolic equation

$$\begin{cases} \partial_t w^\varepsilon(t,x) = \dfrac{1}{2}\partial^2_{xx} w^\varepsilon(t,x) + g'_\varepsilon(w^\varepsilon(t,x)) \cdot \partial_x w^\varepsilon(t,x), \\ w^\varepsilon(0,x) = \varphi'(x). \end{cases} \tag{3.14}$$

By standard theory of parabolic equations,

$$u^\varepsilon \to u, \quad \text{as } \varepsilon \to 0,$$

where u is the unique weak solution to the initial problem of the parabolic equation (3.10), that is the case when $\varepsilon = 0$ for problem (3.13).

Note that g'_ε are uniformly bounded by $|k|$, which is crucial in our argument below. By Nash's continuity theory (see [16]), solutions $\{w^\varepsilon(t,x), \varepsilon > 0\}$ are uniformly Hölder continuous in any compact subset of $(0,\infty)\times\mathbb{R}$, and bounded in $L^2([0,T], H^1_{\text{loc}})$ (where H^1 is the usual Sobolev space). Thus, we can extract a sequence $\varepsilon_n \downarrow 0$, such that $w^{\varepsilon_n}(t,x)$ converges to $w(t,x)$ pointwise, uniform in any compact subset of $(0,T]\times\mathbb{R}$, and w^{ε_n} converges weakly to w in $L^2([0,T], H^1_{\text{loc}})$. For a proof of this basic fact from the theory of parabolic equations, see Lemma 3.2 below.

Since w^ε is a strong solution to problem (3.14), we have for every $\rho(x,t)$ with a compact support in $[0,T)\times\mathbb{R}$,

$$\begin{aligned} -\int_{\mathbb{R}} \rho(x,0)\varphi'(x) = & -\frac{1}{2}\int_{\mathbb{R}\times[0,T)} \partial_x\rho(x,t)\partial_x w^\varepsilon(t,x) \\ & + \int_{\mathbb{R}\times[0,T)} \rho(x,t) g'_\varepsilon(\partial_x u^\varepsilon(t,x)) \cdot \partial_x w^\varepsilon(t,x). \end{aligned}$$

Letting $\varepsilon \to 0$, we therefore obtain that

$$\begin{aligned} -\int_{\mathbb{R}} \rho(x,0)\varphi'(x) = & -\frac{1}{2}\int_{\mathbb{R}\times[0,T)} \partial_x\rho(x,t)\partial_x w(t,x) \\ & + \int_{\mathbb{R}\times[0,T)} \rho(x,t) k \cdot \text{sgn}(w(t,x)) \cdot \partial_x w(t,x), \end{aligned}$$

which implies $w(t,x)$ is the unique weak solution to the problem (3.12).

For every n, according to Itô's formula

$$Y^{\varepsilon_n}_t = \varphi(B_T) + \int_t^T g_{\varepsilon_n}(Z^{\varepsilon_n})\mathrm{d}s - \int_t^T Z^{\varepsilon_n}_s \mathrm{d}B_s, \tag{3.15}$$

where $Z^{\varepsilon_n}_t = w^{\varepsilon_n}(T-t, B.)$ and $Y^{\varepsilon_n}_t = u^{\varepsilon_n}(T-t, B_t)$. Since $w^{\varepsilon_n}(T-t, B.) \to w(T-t, B.)$ and $u^{\varepsilon_n}(T-t,B_t) \to u(T-t,B_t)$ as $n\to\infty$, we have that $Y_t = u(T-t,B_t)$ and $Z_t = w(T-t,B_t)$ are unique solutions to BSDEs:

$$Y_t = \varphi(B_T) + \int_t^T k|Z_s|\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s \qquad \text{for } 0 \le t \le T.$$

Since $\varphi \in C^3(\mathbb{R})$ has polynomial growth, we can apply Lemma 3.1 to $u^\varepsilon(t,x)$. Thus, for each

$\varepsilon > 0$, $w^\varepsilon(t,c) = 0$ for all $t > 0$ and

$$w^\varepsilon(t,x) = \mathbb{E}\left[N_t^\varepsilon \varphi'(B_t + x) \cdot I_{\{t<\tau\}}\right], \tag{3.16}$$

where $\tau = \inf\{s \geq 0 : B_s + x = c\}$ and

$$N_t^\varepsilon = \exp\left[\int_0^t g_\varepsilon'(w^\varepsilon(t-s, B_s + x))\mathrm{d}B_s - \frac{1}{2}\int_0^t |g_\varepsilon'(w^\varepsilon(t-s, B_s+x))|^2 \mathrm{d}s\right].$$

According to the Lebesgue's dominated convergence theorem,

$$\mathbb{E}\left[N_t^{\varepsilon_n}\varphi'(B_t + x) \cdot I_{\{t<\tau\}}\right] \to \mathbb{E}\left[N_t\varphi'(B_t + x) \cdot I_{\{t<\tau\}}\right],$$

as $n \to \infty$. Thus, we conclude that

$$w(t,x) = \mathbb{E}\left[N_t\varphi'(B_t + x) \cdot I_{\{t<\tau\}}\right].$$

The proof is complete. □

In the proof of Theorem 3.3, we use a basic result about parabolic equations. For completeness we provide a proof in the following lemma.

Lemma 3.2 *The linear parabolic equation* (3.12) *has a unique weak solution which is Hölder continuous in any compact set of* $(0,T) \times \mathbb{R}$.

Proof We extract a sequence $\varepsilon_n \downarrow 0$, so that $w^{\varepsilon_n}(t,x) \to w(t,x)$ as $n \to \infty$, where $w(t,x)$ is the unique weak solution to the problem (3.12).

For simplicity, define $b_\varepsilon(t,x) := g_\varepsilon'(\partial_x u^\varepsilon(t,x))$ for every $\varepsilon > 0$, and $b_0(t,x) := k \cdot \mathrm{sgn}(\partial_x u(t,x))$. Then $|b_\varepsilon(t,x)| \leq |k|$, which means that $|b_\varepsilon(t,x)|$ has a bound independent of ε. According to Nash [16] and Aronson [1], the fundamental solution $p_\varepsilon(s,x,t,y)$ to the parabolic equation

$$\left(\partial_t - \frac{1}{2}\partial_{xx} - b_\varepsilon(t,x)\partial_x\right) v = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R} \tag{3.17}$$

is jointly α-Hölder continuous for some α depending only on $|k|$ (see page 328 in Friedman [11] or Nash [16]), and $p_\varepsilon(s,x,t,y)$ has a Gaussian lower and upper bounds uniformly in $\varepsilon \geq 0$, for $0 \leq s < t \leq T$, $x, y \in \mathbb{R}$. This implies that $p_\varepsilon(s,x,t,y)$ is α-Hölder continuous in all its arguments, where α and the Hölder constant are independent of $\varepsilon > 0$. By Aronson [1], the unique weak solution $w^\varepsilon(t,x)$ to the problem (3.14) (for all $\varepsilon \geq 0$) has the representation

$$w^\varepsilon(t,x) = \int_{\mathbb{R}} p_\varepsilon(0,x,t,y)\varphi'(y)\mathrm{d}y. \tag{3.18}$$

Note that $\{p_\varepsilon(s,x,t,y), \varepsilon \geq 0\}$ is a family of equi-continuous functions on any compact set of $0 \leq s < t$ and $x, y \in \mathbb{R}$. Hence, by extracting a sequence ε_n such that $p_{\varepsilon_n}(s,x,t,y)$ converges to $p(s,x,t,y)$ uniformly on any compact subset of $\{0 \leq s < t\} \times \mathbb{R}^2$. Therefore,

$$w^{\varepsilon_n}(t,x) \to w(t,x) \quad \text{uniformly on any compact subset of } (0,\infty) \times \mathbb{R}.$$

Thus, we conclude that $p(s,x,t,y) = p_0(s,x,t,y)$ and $w(t,x) = w^0(t,x)$. □

Thanks to Theorem 3.3, we can determine the sign of (Z_t) for BSDE (1.2) quantitatively.

Corollary 3.1 *Suppose that* $\varphi \in C^3(\mathbb{R})$ *satisfies (H.1) with some constant* c, *such that* (φ, φ') *has at most polynomial growth. Let* (Y_t, Z_t) *be the unique solution pair of the BSDE* (1.2).

(1) *If* $\varphi'(x) \geq 0$ *and* $\varphi'(x) \not\equiv 0$ *for all* $x > c$, *then*

$$\mathrm{sgn}(Z_t) = \mathrm{sgn}(B_t - c), \quad t \geq 0 \textit{ almost surely}.$$

(2) *If* $\varphi'(x) \leq 0$ *and* $\varphi'(x) \not\equiv 0$ *for all* $x > c$, *then*

$$\operatorname{sgn}(-Z_t) = \operatorname{sgn}(B_t - c), \quad t \geq 0 \text{ almost surely.}$$

Proof Similar to the proof of Theorem 3.1, the conclusion follows from Theorem 3.3 directly. □

3.2 Explicit solutions for some BSDEs

Here, we show that Corollary 3.1 allows us to work out the explicit solution of BSDE (1.2), especially when $\xi = I_{[a,b]}(B_T)$ for any two finite constants a, b. To this end, a key technique that we use in the proof is the joint distribution $P(B_t \in \mathrm{d}x, L_t^{\ell} \in \mathrm{d}y)$ of B_t and its local time L_t^{ℓ} with respect to ℓ given by

$$\begin{aligned} &P(B_t \in \mathrm{d}x, L_t^{\ell} \in \mathrm{d}y) \\ &= \frac{1}{\sqrt{2\pi t^3}}(y + |x-\ell| + |\ell|) \exp\left\{\frac{-(y+|x-\ell|+|\ell|)^2}{2t}\right\} \cdot I_{\{y>0\}} \mathrm{d}x\mathrm{d}y \\ &\quad + \frac{1}{\sqrt{2\pi t}}\left[\exp\left\{-\frac{x^2}{2t}\right\} - \exp\left\{-\frac{(|x-\ell|+|\ell|)^2}{2t}\right\}\right] \cdot I_{\{y=0\}} \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{3.19}$$

(see [3], for example).

Theorem 3.4 *Suppose that* $\varphi \in C^3(\mathbb{R})$ *satisfies (H.1) and* $(\varphi, \ \varphi')$ *has at most polynomial growth. Then the unique solution pair of BSDE* (1.2) *is given by*

$$Y_t = H(B_t), \quad Z_t = \partial_h H(B_t), \tag{3.20}$$

where $H(h)$ *is defined in the following.*

(i) *If* $\varphi' \geq 0$ *and* $\varphi' \not\equiv 0$ *on* (c, ∞), *then*

$$\begin{aligned} &H(h) = e^{-\frac{1}{2}k^2(T-t)} \times \\ &\left\{\int_{\mathbb{R}}\int_{y\geq 0} \varphi(x+h)e^{k|x-c+h|-k|c-h|-ky} \cdot P(B_{T-t} \in \mathrm{d}x, L_{T-t}^{c-h} \in \mathrm{d}y)\right\}. \end{aligned} \tag{3.21}$$

(ii) *If* $\varphi' \leq 0$ *and* $\varphi' \not\equiv 0$ *on* (c, ∞), *then*

$$\begin{aligned} &H(h) = e^{-\frac{1}{2}k^2(T-t)} \times \\ &\left\{\int_{\mathbb{R}}\int_{y\geq 0} \varphi(x+h)e^{-k|x-c+h|+k|c-h|+ky} \cdot P(B_{T-t} \in \mathrm{d}x, L_{T-t}^{c-h} \in \mathrm{d}y)\right\}. \end{aligned} \tag{3.22}$$

Proof (i) Since $\varphi' \geq 0$ and $\varphi' \not\equiv 0$ on (c, ∞), we have $\operatorname{sgn}(Z_t) = \operatorname{sgn}(B_t - c)$ by Corollary 3.1. Then BSDE (1.2) can be rewritten as linear BSDEs in Z

$$Y_t = \varphi(B_T) + k\int_t^T \operatorname{sgn}(B_s - c)Z_s \mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s.$$

By Girsanov's theorem, Y_t is given by

$$Y_t = \mathbb{E}\left[\varphi(B_T) \cdot e^{-\frac{1}{2}k^2(T-t)+k\int_t^T \operatorname{sgn}(B_s-c)\mathrm{d}B_s}\middle|\mathcal{F}_t\right]. \tag{3.23}$$

According to Tanaka's formula, we have

$$\int_0^T \operatorname{sgn}(B_s - c)\mathrm{d}B_s = |B_T - c| - |c| - L_T^c.$$

Then, when $t=0$, Y_t in (3.23) can be solved by

$$\begin{aligned} Y_0 &= \mathbb{E}_P\left[\varphi(B_T)e^{-\frac{1}{2}k^2T+k\int_0^T \operatorname{sgn}(B_s-c)\mathrm{d}B_s}\right] \\ &= \mathbb{E}_P\left[\varphi(B_T)\cdot e^{-\frac{1}{2}k^2T+k(|B_T-c|-|c|-L_T^c)}\right] \\ &= e^{-\frac{1}{2}k^2T}\int_{\mathbb{R}}\int_{y\geq 0}\varphi(x)\exp\{k|x-c|-k|c|-ky\}\cdot P(B_T\in \mathrm{d}x, L_T^c\in \mathrm{d}y). \end{aligned} \tag{3.24}$$

Therefore, Y_t is given by

$$\begin{aligned} Y_t = e^{-\frac{1}{2}k^2(T-t)}&\int_{\mathbb{R}}\int_{y\geq 0}\varphi(x+h)\exp\{k|x-c+h|-k|c-h|-ky\}\times \\ &P(B_{T-t}\in \mathrm{d}x, L_{T-t}^{c-h}\in \mathrm{d}y)|_{h=B_t} \\ =: &\, H(B_t). \end{aligned}$$

Then by Theorem 3.3, we have $Z_t=\partial_h H(B_t)$.

(ii) Similarly, if $\varphi'\leq 0$ and $\varphi'\not\equiv 0$ on (c,∞), by Corollary 3.1 we have $\operatorname{sgn}(Z_t)=-\operatorname{sgn}(B_t-c)$. The rest can be proved in a similar manner as for **(i)**. The proof is completed. □

Thanks to Theorem 3.4, we are able to work out a closed-form of solutions (Y_t,Z_t) to the following BSDE,

$$Y_t = I_{[a,b)}(B_T)+\int_t^T k|Z_s|\mathrm{d}s-\int_t^T Z_s\mathrm{d}B_s, \tag{3.25}$$

where a,b are finite constants.

Corollary 3.2 *Let Φ be the cumulative distribution function of standard normal distribution. Set $c:=\dfrac{a+b}{2}$ for any $a,b\in(-\infty,\infty)$. Then the explicit solutions of BSDE (3.25) are given as:*

$$\begin{aligned} Y_t = &\,\Phi\left(-\frac{|B_t-c|-k(T-t)-\frac{b-a}{2}}{\sqrt{T-t}}\right) \\ &- e^{-k(b-a)}\cdot\Phi\left(-\frac{|B_t-c|-k(T-t)+\frac{b-a}{2}}{\sqrt{T-t}}\right), \end{aligned} \tag{3.26}$$

and

$$Z_t = \frac{-\operatorname{sgn}(B_t-c)}{\sqrt{2\pi(T-t)}}\left\{e^{-\frac{[|B_t-c|-k(T-t)-\frac{b-a}{2}]^2}{2(T-t)}}-e^{-k(b-a)}e^{-\frac{[|B_t-c|-k(T-t)+\frac{b-a}{2}]^2}{2(T-t)}}\right\}. \tag{3.27}$$

Proof For any $\varepsilon>0$, let us set

$$\varphi_\varepsilon(x)=\mathbb{E}[I_{[a,b)}(x+\sqrt{\varepsilon}\xi)]=\int_{-\infty}^{\infty}I_{[a,b]}(v)\frac{1}{\sqrt{2\pi\varepsilon}}\exp\left[-\frac{(v-x)^2}{2\varepsilon}\right]\mathrm{d}v,$$

where ξ is a standard normal distribution under probability measure P. Then $\varphi_\varepsilon\in C^\infty(\mathbb{R})$ and $\varphi_\varepsilon(x)\to I_{[a,b)}(x)$ as $\varepsilon\to 0$.

Consider the following BSDE

$$Y_t^\varepsilon=\varphi_\varepsilon(B_T)+\int_t^T k|Z_s|\mathrm{d}s-\int_t^T Z_s\mathrm{d}B_s.$$

Since $\varphi_\varepsilon(x) \in C^\infty(\mathbb{R})$, by Theorem 3.4 we know Y_t^ε is given by

$$\begin{aligned} Y_t^\varepsilon &= e^{-\frac{1}{2}k^2(T-t)}\left\{ \int_{\mathbb{R}}\int_{y\geq 0} \varphi_\varepsilon(x+h)e^{k|x-c+h|-k|c-h|-ky}\times \right.\\ &\qquad \left. P\left(B_{T-t}\in \mathrm{d}x, L_{T-t}^{c-h}\in \mathrm{d}y\right)\right\}\Big|_{h=B_t} \\ &= e^{-\frac{1}{2}k^2(T-t)}\left[\int_{-\infty}^{\infty}\varphi_\varepsilon(x+h)\cdot f(c,h,x)\mathrm{d}x\right]\Bigg|_{h=B_t} =: H_\varepsilon(h)|_{h=B_t}, \end{aligned}$$

where

$$f(c,h,x) = \int_0^\infty \exp\{-k|x-c+h|+k|c-h|+ky\}f_1(c,h,x,y)\mathrm{d}y + f_2(c,h,x).$$

In the above equation, f_1 and f_2 are given by

$$f_1(c,h,x,y) = \frac{y+|x-(c-h)|+|c-h|}{\sqrt{2\pi(T-t)^3}}\exp\left[-\frac{(y+|x-(c-h)|+|c-h|)^2}{2(T-t)}\right],$$

and

$$f_2(c,h,x) = \frac{e^{-k|x-c+h|+k|c-h|}}{\sqrt{2\pi(T-t)}}\left\{e^{-\frac{x^2}{2(T-t)}} - \exp\left[-\frac{(|x-(c-h)|+|c-h|)^2}{2(T-t)}\right]\right\}.$$

Now, we prove that

$$H_\varepsilon(h) \to H(h) \quad \text{as } \varepsilon \to 0,$$

where

$$H(h) = \Phi\left(-\frac{|h-c|-k(T-t)-\frac{b-a}{2}}{\sqrt{T-t}}\right) - e^{-k(b-a)}\Phi\left(-\frac{|h-c|-k(T-t)+\frac{b-a}{2}}{\sqrt{T-t}}\right).$$

Actually, we have

$$\begin{aligned} H(h) &= e^{-\frac{1}{2}k^2(T-t)}\left\{ \int_{\mathbb{R}}\int_{y\geq 0} I_{[a,b)}(x+h)e^{k|x-c+h|-k|c-h|-ky}\times \right.\\ &\qquad \left. P(B_{T-t}\in \mathrm{d}x, L_{T-t}^{c-h}\in \mathrm{d}y)\right\} \\ &= e^{-\frac{1}{2}k^2(T-t)}\left[\int_{-\infty}^{\infty} I_{[a,b)}(x+h)\cdot f(c,h,x)\mathrm{d}x\right]. \end{aligned}$$

By Lebesgue's dominated convergence theorem, $H_\varepsilon(h)$ converges to $H(h)$ as $\varepsilon \to 0$, which means that $H_\varepsilon(B_t)$ converges to $H(B_t)$ almost surely. Therefore,

$$\begin{aligned} Y_t &= H(B_t) \\ &= \Phi\left(-\frac{|B_t-c|-k(T-t)-\frac{b-a}{2}}{\sqrt{T-t}}\right) - e^{-k(b-a)}\Phi\left(-\frac{|B_t-c|-k(T-t)+\frac{b-a}{2}}{\sqrt{T-t}}\right). \end{aligned}$$

According to El Karoui, Peng, and Quenez [17, Corollary 4.1], we have $Z_t = \partial_h H(B_t)$. Thus, Z_t is given by (3.27), from which it is not hard to see that

$$\mathrm{sgn}(Z_t) = -\mathrm{sgn}(B_t - c),$$

which means Theorem 3.1 also holds for indicator function.

The proof is complete. □

Remark 3.6 *We point out that Corollary 3.2 is also applicable when* $a=-\infty$ *or* $b=+\infty$. *For example,* $\xi=I_{\{B_T<b\}}$ *if* $a=-\infty$. *Then* $c=\dfrac{a+b}{2}=-\infty$ *for any* $b\in\mathbb{R}$. *By Theorem 3.3,* $\mathrm{sgn}(Z_t)=-\mathrm{sgn}(B_t-c)=-1$, *which implies that* $Z_t<0$ *for* $t\in[0,T]$. *Thus, BSDE* (3.25) *becomes a linear BSDE in this case:*

$$Y_t=I_{\{B_T\le b\}}-\int_t^T kZ_s\mathrm{d}s-\int_t^T Z_s\mathrm{d}B_s.$$

By Girsanov's formula and the relation between Y_t *and* Z_t, *we can obtain the explicit solutions easily.*

Remark 3.7 *We plot one sample path of Brownian motion* B_t *and the solution* Z_t *of Corollary 3.2 in Figure 1, in which the blue line is* B_t *and the red is* Z_t. *We can see the relationship of the sign between* B_t-c *and* Z_t *intuitively in this figure.*

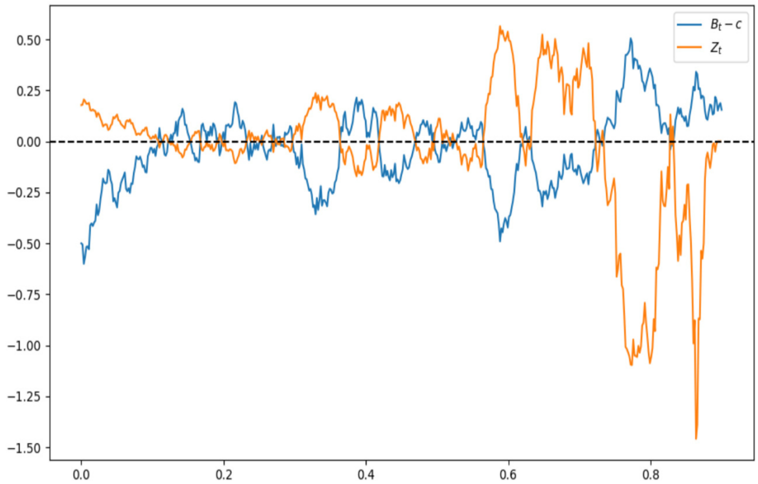

Figure 1 Brownian motion B_t (blue) and solution Z_t (red) ($a=0,\ b=1,\ k=0.1,\ T=0.9$)

4. Applications

In this section, as an application of Theorem 3.4, we give two examples of explicit solutions. One is BSDE (1.2) when $\varphi(x)=x^2$, the other is the following PDE:

$$\begin{cases}\partial_t u(t,x)=\dfrac{1}{2}\partial_{xx}^2u(t,x)+k\cdot\mathrm{sgn}(x)\partial_x u(t,x),\\ \lim\limits_{t\to 0^+}u(t,x)=x^2.\end{cases}\tag{4.1}$$

Additionally, we apply the explicit solution of BSDE (3.25) in Corollary 3.2 to the contingent pricing in an incomplete market.

4.1 Applications to BSDEs and PDE

Example 4.1 *The explicit solution pair* (Y_t,Z_t) *of BSDE*

$$Y_t=B_T^2+k\int_t^T|Z_s|\mathrm{d}s-\int_t^T Z_s\mathrm{d}B_s,\tag{4.2}$$

is given by

$$Y_t = \frac{1}{2k^2} + \sqrt{\frac{T-t}{2\pi}}\left[|B_t| + k(T-t) + \frac{1}{k}\right]\exp\left\{-\frac{[|B_t|+k(T-t)]^2}{2(T-t)}\right\}$$
$$+\left\{[|B_t|+k(T-t)]^2 + (T-t) - \frac{1}{2k^2}\right\}\Phi\left(\frac{|B_t|+k(T-t)}{\sqrt{T-t}}\right)$$
$$+e^{-2k|B_t|}(|B_t| + T - t - \frac{1}{2k^2})\Phi\left(-\frac{|B_t|-k(T-t)}{\sqrt{T-t}}\right) \tag{4.3}$$

and

$$Z_t = \sqrt{\frac{T-t}{2\pi}}\cdot \mathrm{sgn}(B_t)\cdot \exp\left\{-\frac{[|B_t|+k(T-t)]^2}{2(T-t)}\right\}$$
$$\times\left\{1 + \left[|B_t| + k(T-t) + \frac{1}{k}\right]\cdot\left[-\frac{[|B_t|+k(T-t)]}{(T-t)}\right]\right\}$$
$$+2\mathrm{sgn}(B_t)\cdot[|B_t|+k(T-t)]\cdot\Phi\left(\frac{|B_t|+k(T-t)}{\sqrt{T-t}}\right)$$
$$+\frac{\mathrm{sgn}(B_t)}{\sqrt{2\pi(T-t)}}\exp\left\{-\frac{[|B_t|+k(T-t)]^2}{2(T-t)}\right\}$$
$$\times\left\{[|B_t|+k(T-t)]^2 - k(T-t) - \frac{1}{2k^2}\right\}$$
$$+e^{-2k|B_t|}\mathrm{sgn}(B_t)\Phi\left(-\frac{|B_t|-k(T-t)}{\sqrt{T-t}}\right)\left[-2k\left(|B_t| + T - t - \frac{1}{2k^2}\right) + 1\right]$$
$$-e^{-2k|B_t|}\left(|B_t| + T - t - \frac{1}{2k^2}\right)\frac{\mathrm{sgn}(B_t)}{\sqrt{2\pi(T-t)}}\exp\left\{-\frac{[|B_t|-k(T-t)]^2}{2(T-t)}\right\}. \tag{4.4}$$

Proof By Theorem 3.4,

$$Y_t = e^{-\frac{1}{2}k^2(T-t)}\left\{\int_{\mathbb{R}}\int_{y\geq 0}(x+h)^2 e^{k|x+h|-k|h|-ky}\cdot P(B_{T-t}\in \mathrm{d}x, L_{T-t}^{-h}\in \mathrm{d}y)\right\}\Bigg|_{h=B_t}.$$

By an elementary calculation, we have

$$Y_t = \frac{1}{2k^2} + \sqrt{\frac{T-t}{2\pi}}\left[|B_t| + k(T-t) + \frac{1}{k}\right]\exp\left\{-\frac{[|B_t|+k(T-t)]^2}{2(T-t)}\right\}$$
$$+\left\{[|B_t|+k(T-t)]^2 + (T-t) - \frac{1}{2k^2}\right\}\Phi\left(\frac{|B_t|+k(T-t)}{\sqrt{T-t}}\right)$$
$$+e^{-2k|B_t|}(|B_t| + T - t - \frac{1}{2k^2})\Phi\left(-\frac{|B_t|-k(T-t)}{\sqrt{T-t}}\right). \tag{4.5}$$

Therefore, by Theorem 3.3, we have $Z_t = \partial_h H(B_t)$. Thus, we obtain Z_t as equation (4.4). The proof is completed. □

It is interesting that our result can be used to get the explicit solution of PDE (4.1), its application can be found in [12] for $k = -1$.

Example 4.2 *The unique weak solution to the initial problem of parabolic equation* (4.1) *is given by*

$$u(t,x) = \frac{1}{2k^2} + \sqrt{\frac{t}{2\pi}}(|x| + kt + \frac{1}{k}) \exp\left\{ - \frac{(|x|+kt)^2}{2t} \right\}$$
$$+ \left\{ (|x|+kt)^2 + t - \frac{1}{2k^2} \right\} \Phi\left(\frac{|x|+kt}{\sqrt{t}} \right)$$
$$+ e^{-2k|x|}(|x| + T - t - \frac{1}{2k^2}) \Phi\left(- \frac{|x|-kt}{\sqrt{t}} \right).$$

The explicit solution agrees with the result of [12] (see Exercise 5.3 on Page 441) when $k=-1$.

Proof By Theorem 3.3, $(Y_t, Z_t) = (u(T-t, B_t), \partial_x u(T-t, B_t))$ is the unique solution to the BSDE

$$Y_t = B_T^2 + k \int_t^T |Z_s| \mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s.$$

By Theorem 3.1, $\mathrm{sgn}(B_t) = \mathrm{sgn}(Z_t)$ for $\varphi(x) = x^2$. Hence the expression for $u(t,x)$ follows from (4.3) immediately. □

4.2 Robust prices in incomplete markets

The Black-Scholes model studied by Black and Scholes (1973), Merton (1973, 1991) is the most celebrated example of option pricing and hedging in a complete market using no-arbitrage theory and martingale methods. According to this theory, when a stock obeys the geometric Brownian motion

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t, \quad S_0 = 1, \tag{4.6}$$

there exists a unique risk-neutral martingale measure Q such that the price of the contingent claim ξ at time T is given by $\mathbb{E}_Q[\xi e^{-rT}]$, where

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_T} = e^{\int_0^T (\frac{\mu - r}{\sigma}) \mathrm{d}B_s - \frac{1}{2}\int_0^T (\frac{\mu-r}{\sigma})^2 \mathrm{d}s}, \tag{4.7}$$

where r is the interest rate of a bond. Therefore, for $\xi = I_{\{a \le S_T \le b\}}$, the price of the contingent claim ξ is given by

$$\mathbb{E}_Q\left(\xi e^{-rT}\right) = e^{-rT}\left[\Phi\left(\frac{\ln b - (2\mu - r - 0.5\sigma^2)T}{\sigma\sqrt{T}} \right) - \Phi\left(\frac{\ln a - (2\mu - r - 0.5\sigma^2)T}{\sigma\sqrt{T}} \right) \right]. \tag{4.8}$$

In an incomplete market, the incompleteness of the market usually gives rise to infinitely many martingale measures, therefore upper and lower pricing was studied by El Karoui and Quenez (1995) [9] and El Karoui and Peng (1997) [10]. They use min-max pricing to show that the pricing of an insurance or contingent claim equals the maximal (minimal) expectations with respect to a set of martingale measures. Chen and Epstein (2002) studied ambiguity pricing under a set of measures $\mathcal{P}$, where

$$\mathcal{P} = \left\{ Q : \left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \exp\left[\int_0^t \theta_s \mathrm{d}B_s - \frac{1}{2}\int_0^t \theta_s^2 \mathrm{d}s \right], \quad \sup_{s\in[0,T]} |\theta_s| \le k \right\}. \tag{4.9}$$

Set

$$y_t = \operatorname*{ess\,inf}_{Q\in\mathcal{P}} \mathbb{E}_Q[\xi|\mathcal{F}_t] \ \text{ and } \ Y_t = \operatorname*{ess\,sup}_{Q\in\mathcal{P}} \mathbb{E}_Q[\xi|\mathcal{F}_t]. \tag{4.10}$$

It is known that y_t and Y_t are, respectively, the minimum and maximum price of ξ in an incomplete market.

Chen and Epstein (2002)[5] have shown that there exists an adapted z_t such that Y_t and y_t are expressed as

$$y_t = \xi - k\int_t^T |z_s|\mathrm{d}s - \int_t^T z_s \mathrm{d}B_s \tag{4.11}$$

and

$$Y_t = \xi + k\int_t^T |z_s|\mathrm{d}s - \int_t^T z_s \mathrm{d}B_s. \tag{4.12}$$

Using our results from the previous sections, we give the explicit representation of the wealth Y_t when the stock price S_t obeys the geometric Brownian motion

$$S_t = \exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right] \tag{4.13}$$

and $\xi = I_{\{a\le S_T\le b\}}$.

In fact, when $\xi = I_{\{a\le S_T\le b\}}$, that is,

$$\xi = I_{\left\{\frac{\ln a-(\mu-0.5\sigma^2)T}{\sigma}\le B_T\le \frac{\ln b-(\mu-0.5\sigma^2)T}{\sigma}\right\}}. \tag{4.14}$$

According to the calculation in Corollary 3.2, with $c = \dfrac{\ln(ab)}{2\sigma} - \dfrac{(\mu-0.5\sigma^2)T}{\sigma}$, we have the upper pricing, which is given by

$$Y_t = \Phi\left(-\frac{|B_t - c| - k(T-t) - \frac{\ln(b/a)}{2\sigma}}{\sqrt{T-t}}\right) - e^{-k\frac{\ln(b/a)}{\sigma}}\Phi\left(-\frac{|B_t - c| - k(T-t) + \frac{\ln(b/a)}{2\sigma}}{\sqrt{T-t}}\right), \tag{4.15}$$

and the lower pricing is given by

$$y_t = \Phi\left(-\frac{|B_t - c| + k(T-t) - \frac{\ln(b/a)}{2\sigma}}{\sqrt{T-t}}\right) - e^{k\frac{\ln(b/a)}{\sigma}}\Phi\left(-\frac{|B_t - c| + k(T-t) + \frac{\ln(b/a)}{2\sigma}}{\sqrt{T-t}}\right). \tag{4.16}$$

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